Some Chaotic Properties on -space

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الخلاصة

درسنا بعض الخواص الفوضوية في فضاء-G، برهنا إذا كانت الدالة G-localy eventually فإنها تمتلك حساسية معتمدة على الشروط الابتدائية في فضاء-G، أما إذا كانت الدالة تمتلك حساسية متسلسلة في فضاء-G وكانت f ترافق h فإن h أيضا تملك حساسية متسلسلة. أخيرا برهنا بعض الخواص الفوضوية التي تخص الخلط-G.

الكلمات المفتاحية

الخواص الفوضوية، الدالة متعددة الحساسية، الدالة ذات الحساسية المتسلسلة.

Abstract

We study some chaotic properties on G-space, we prove if f is G-localy eventually onto then f is G-sensitive dependence on initial conditions, also if f is G-chain sensitive then is also G-chain sensitive when $f \approx h$.Also we generalize some properties about G-blending.

Key words

Chaotic properties, Sensitive dependence function, Chain sensitive function.

1. Introduction

In this paper, we introduce some chaotic properties on G-space as; G-sensitive dependence on initial conditions, G-localy eventually onto and G-blending (strongly and weakly). We know that sensitive dependence on initial conditions has played an important role in the development of the theory of chaotic dynamical system. The goal of this research is to give definition of G-chain sensitive in the case when X is a compact G-space and we prove if f is G-chain sensitive then h is also G-chain sensitive when $f \approx h$. In [1] Iftichar Al-shara`a and May Al-Yaseen proved that if $f$ is locally eventually and onto then $f$ has sensitive dependence on initial conditions, we find that this result is also satisfied on $G$-space. Another important dynamical property is blending (weakly and strongly). In [2] Iftichar Al-Shara`a showed that $f:X \to X$ and $g:Y \to Y$ are weakly blending maps if and only if $f \times g: X \times Y \to X \times Y$ is weakly blending map and some results on strongly blending, we generalize these results on G-space. One such setting is the study of these properties on G-space. In section 2, we give some definitions and preliminaries. In section 3, we prove that G-locally eventuality and surjection and implies G-sensitive dependence on initial conditions and some other results on G-blending.

2. Preliminaries

Let Z denote the set of integers and N denote the set of natural numbers. By a G-space, we mean a triple (X, G, θ), where X is a Hausdorff space, G is a topological group and $\theta: G \times X \to X$ is a continuous action of G on X. For $x \in X$, the set $G(x) = \{\theta(g, x): g \in G\}$, is called the G-orbit of x in X. Note that G-orbits $G(x)$ and $G(y)$ of points $x, y \in X$ are either disjoint or equal. If X, Y are G-spaces, then a continuous map $h: X \to Y$ is called equivariant if $h(\theta(g, x)) = \theta(g, h(x))$, for each g in G and each x in X [3]. We will be denoted of $\theta(g, x)$ by gx. Now, we give some definitions on G-space. A map $f: X \to X$ is called ***G*-sensitive** at $x \in X$ if for all open set U containing x there exists $g_1, g_2 \in G$ and there exists $y \in U$ with $y \notin G(x)$ there is $n \in N$ and V open subset of X such that $g_1 . f^n(x) \in V$ and $g_2 . f^n(y) \in \overline{V}$. A map f is ***G*-locally eventually onto** if for every nonempty subset U of X there exists a positive integer n_o and $g \in G$ such that for every $n \geq n_o$, $g. f^n(U) = X$. Let (G_1, X, θ_1) and (G_2, Y, θ_2) be two transformation groups. Two maps $f: X \to X$ and $h: Y \to Y$ are said to be **equivariant topologically conjugate** if there exists an isomorphism $(\mu, \varphi):(G_1, X, \theta_1) \to (G_2, Y, \theta_2)$ such that φ is topological conjugate, that is, $\varphi f = h\varphi$. In this case we say that f, h are equivariant topologically conjugate [3].Let $f: X \to X$ be a continuous map on a metric G-space (X, d) and δ be a positive real number, a sequence $\{x_n\}_{n \in N}$ in X is said to be ***G*-δchain** if there exists $g_n \in G$ such that $d(g_n . f(x_n), x_{n+1}) < \delta$ for all n [4]. A map $f: X \to X$ is called ***G*-weakly blending**, if for any pair of nonempty open sets U and V in X there is some $n > 0$ and $g_1, g_2 \in G$ so that $g_1 . f n(U) \cap g_2 . f^n(V) \neq \emptyset$. A map $f: X \to X$ is called ***G*-strongly blending**, if for all nonempty open sets $U, V \subset X$ there is some $n > 0$ and $g_1, g_2 \in G$ so that $g_1 . f^n(U) \cap g_2 . f^n(V) = W$; where W is open subset of X. A point $x \in X$ is said to be ***G*-periodic point** of f if there exists $n \in N$ and $g \in G$ such that $g. f^n(x) = x$......(2. 1). The smallest positive integer n satisfying (2. 1) is called period of f at x. Let $f: X \to X$ be a continuous map on a

G-space X, we say f satisfy ***G*-Touhey property** if for all nonempty open sets U, $V \subset X$, there is a G-periodic point $p \in U$ and a non negative integer k and $g \in G$ such that $g.\, f^{k}(p) \in V$, that is, every pair of nonempty open subsets of X shares a G-periodic orbit. Let $\{f_n\} n \in N$ be a sequence of continuous self-maps on a metric G-space (X, d). we say that $\{f_n\}$ is ***G*-orbitally convergent** to a map $f{:}X \to X$ if for every $\epsilon > 0$, there exists $p \in N$ such that $d\,(g.\, f_n^{m}(x), k.\, f^{m}(x)) < \epsilon$ for all $x \in X$, for all $m \in N$, for all $g, k \in G$ and for all $n \geq p$ [4].

3. Main Theorems

In this section, we prove some results on G-sensitivity and G-blending.

3. 1. Theorem

Let X be a G-space and $f{:}X \to X$ be a map. If f is G-locally eventually and onto then f is G-sensitive dependence on initial conditions.

Proof:

Suppose that A and B are two subsets of X such that $A \cap B \neq \emptyset$ take $x \in X$ and choose V be any open subset of X such that $x \in V$. Since f is G-locally eventually and onto then there exist $m_1 \in N$ and $g_1 \in G$ such that $g_1.\, f^{n1}(A) = X; \forall n_1 \geq m_1$, also there exist $m_2 \in N$ and $g_2 \in G$ such that $g_2.\, f^{n2}(B) = X; \forall n_2 \geq m_2$. So we have $m_1 < n_1 \in N$ and $m_2 < n_2 \in N$ such that $g_1.\, f^{m1}(A) \cap V = \emptyset$ and $g_2.\, f^{m2}(B) \cap V = \emptyset$. Let $m = min\{m_1, m_2\}$, then there exists $y_1 \in A$ with $y_1 \notin G(x)$ and there exists $g_3 \in G$ such that $g_3.\, f^{m}(y_1) \in \overline{V}$ and there exists $y_2 \in B$ with $y_2 \notin G(x)$ and there exists $g_4 \in G$ such that $g_4.\, f^{m}(y_2) \notin \overline{V}$, so we have $g_5.f^{m}(x) \in V$ and $g_3.f^{m}(y_1) \notin \overline{V}$ or $g_5.\, f^{m}(x) \in V$ and $g_4.\, f^{m}(y_2) \notin \overline{V}$ for some $g_5 \in G$. So f is G-sensitive dependence on initial conditions.

We define the chain sensitive dependence on initial condition as: A map $f{:}X \to X$ is called ***G*-chain sensitive** at x in x if for any $\delta > 0$, there exists $z \in X$ for all $y \in X$, and there exists finite G-δchain $y_o, \ldots, y_n$ such that $y_o = y$ and $y_n = x$ and there is no finite G-δchain $z_o, \ldots, z_n$ such that $z_o = x$ and $z_n = z$.

3. 2. Theorem

Let $f{:}X \to X$, $h{:}Y \to Y$ be two maps on a metric G-space and let $f \approx h$. If f is G-chain sensitive at $x \in X$ then h is also G-chain sensitive at x.

Proof:

Let $x, y \in X$ and let $\delta > 0$. Since f is G-chain sensitive then there exists $z \in X$ and for any $y \in X$ there is finite G-δchain $y_o, \ldots, y_n$ such that $y_o = y$, $y_n = x$ and there is no finite G-δchain $z_o, \ldots, z_n$ such that $z_o = x$, $z_n = z$. By hypothesis, f conjugate to h then there exists a homeomorphism $\varphi{:}X \to Y$ such that $\varphi_o\, f = h_o\, \varphi$ This implies that for all $\varphi\,(g_o, x)$, $\varphi\,(g_n, y) \in Y$ there is $\mu\,(g_o) = g_o^{*}$, $\mu\,(g_n) = g_n^{*} \in G_2$ and there exists $\varphi\,(g_n, z)$ for all $\varphi\,(g_n, y) \in Y$ there is finite G-δchain $\varphi\,(g_o, y_o), \ldots, \varphi\,(g_n, y_n)$ such that $\varphi\,(g_o, y_o) = \varphi\,(g_o, y) = (g_o^{*}, \varphi\,(y))$, $\varphi\,(g_n, y_n) = \varphi\,(g^{n}, x) = (g_n^{*}, \varphi\,(x))$ and there is no finite G-δchain $\varphi\,(g_o, z_o), \ldots, \varphi\,(g_n, z_n)$ such that $\varphi\,(g_o, z_o) = \varphi\,(g_o, x) = (g_o^{*}, \varphi\,(x))$, $\varphi\,(g_n, z_n) = \varphi\,(g_n, z) = (g_n^{*}, \varphi\,(z))$. So h is G-δchain sensitive.

3. 3. Theorem

Let $\{f_n\}_{n \in N}$ be a sequence of G-strongly blending maps on a metric G-space (X, d) converges G-orbitally to a map f, then f is also G-strongly blending.

Proof:

Let A, B be a nonempty open sets in X. Since $\{f_n\}$ G-orbitally convergent to f, then for all $x \in X$ and for all $\epsilon > 0$ there exists $k \in N$ such that $d\,(g.\, f_n^{m}(x), h.\, f^{m}(x)) < (\epsilon/4)\ \forall n \geq k, \forall g, h \in G$ and $\forall m$

ϵ *N*. Since f_n is *G*-strongly blending, then there is some $m > 0$ and there are $g_1, g_2 \in G$ such that $g_1 . f_n^m(A) \cap g_2 . f_n^m(B) = W$; where *W* is nonempty open set. Since *W* is open then there is $\epsilon > 0$ and $p \in X$ such that $W(p, \epsilon) \subset W$, where $W(p, \epsilon)$ is open ball with center *p* and radius ϵ, which implies $g_1 . f_n^m(A) \cap g_2 . f_n^m(B) = W(p, (\epsilon/4))$. Note that $g_1 . f_n^m(A) \cap W(p, (\epsilon/4)) \neq Ø$ and $g_2 f_n^m(B) \cap W(p, (\epsilon/4)) \neq Ø$, hence there are $a \in A$ and $b \in B$ satisfying $d(g_1 . f_n^m(a), p) < (\epsilon/4)$ and $d(g_2 . f_n^m(b), p) < (\epsilon/4)$, so we have $d(h_1 . f^m(a), h_2 . f^m(b)) \le d(g . f_n^m(a), h_1 . f^m(a)) + d(g . f_n^m(a), h_2 . f^m(b)) \le d(g . f_n^m(a), h_1 . f^m(a)) + d(g . f_n^m(a), h . f_n^m(b)) + d(h . f_n^m(b), h_2 . f^m(b)) \le d(g . f_n^m(a), h_1 . f^m(a) + d(g . f_n^m(a), p) + d(h . f_n^m(b), p) + d(h . f_n^m(b), h_2 . f^m(b)) < (\epsilon/4) + (\epsilon/4) + (\epsilon/4) + (\epsilon/4) = \epsilon$. So that $d(h_1 . f^m(a), p) < \epsilon$ and $d(h_2 . f^m(b), p) < \epsilon$, that mean, $h_1 . f^m(A) \cap W(p, \epsilon) \neq Ø$ and $h_2 . f^m(B) \cap W(p, \epsilon) \neq Ø$ so that $h_1 . f^m(A) \cap h_2 . f^m(B) = W(p, \epsilon) \subset W$ and consequently $h_1 . f^m(A) \cap h_2 . f^m(B) = W$ for some $m > 0$, $h_1, h_2 \in G$. Thus *f* is *G*-strongly blending.

As a direct consequence of the above theorem we have the following corollary:

3. 4. Corollary

If $\{f_n\}_{n \in N}$ is a sequence of *G*-weakly blending maps on a metric *G*-space (X, d) converges *G*-orbitally to a map *f*, then *f* is also *G*-weakly blending.

Proof:

Since each *G*-strongly blending map, is *G*-weakly blending then the corollary holds.

Now, if *f* is G_1-transitive and *h* is G_2-transitive, then $f \times h$ is not Necessarily $G_1 \times G_2$-transitive, but this is holds when the maps are *G*-blending; as the following theorem:

3. 5. Theorem

Let *X* be a G_1-space and *Y* be a G_2-space. f: X→X and h: Y→Y are maps. f is G_1-weakly blending and *h* is G_2-weakly blending if and only if $f \times h: X \times Y \to X \times Y$ is $G_1 \times G_2$-weakly blending.

Proof:

Suppose that f is G_1-weakly blending and h is G_2-weakly blending. Let U, V be two nonempty open subsets of X × Y, then there are U_1 and V_1 nonempty open subsets of X, and there are U_2 and V_2 nonempty open subsets of Y such that $U = U_1 \times U_2$ and $V = V_1 \times V_2$. Since f is G_1-weakly blending, then there exists $k_1 \epsilon N$ and there are $g_1, g_2 \epsilon G_1$ such that $g_1 . f(k_1)(U_1) \cap g_2 . f(k_1)(V_1) \neq Ø$, also since h is G_2-weakly blending, then there exist $k_2 \epsilon N$ and $g_3, g_4 \epsilon G_2$ such that $g_3 . h(k_2)(U_2) \cap g_4 . h(k_2)(V_2) \neq Ø$. Choose $k = k_1 + k_2$, then $g_1 . f^k(U_1) \cap g_2 . f^k(V_1) \neq Ø$ and $g_3 . h^k(U2) \cap g_4 . h^k(V_2) \neq Ø$. Then, we have $((g_1, g_3).(f \times h)^k(U)) \cap ((g_2, g_4).(f \times h)^k(V)) = ((g_1, g_3).(f \times h)^k(U_1 \times U_2)) \cap ((g_2, g_4).(f \times h)^k(V_1 \times V_2)) = (g_1 . f^k(U_1) \times g_3 . h^k(U_2)) \cap (g_2 . f^k(V_1) \times g_4 . h^k(V_2)) = (g_1 . f^k(U_1) \cap g_2 . f^k(V_1)) \times (g_3 . h^k(U_2) \cap g_4 . h^k(V_2)) \neq Ø$.

Therefore, f×h is $G_1 \times G_2$-weakly blending.

Conversely, suppose that f×h is $G_1 \times G_2$-weakly blending, we prove that f is G_1-weakly blending. Let U_1 and V_1 are nonempty open sets in X, thus there exist $U = U_1 \times Y$ and $V = V_1 \times Y$ are nonempty open subsets of X×Y. Since f×h is $G_1 \times G_2$-weakly blending, then there exists kϵN and there is $g_1, g_2 \epsilon G_1$, $g_3, g_4 \epsilon G_2$ such that $((g_1, g_3).(f \times h)^k(U)) \cap ((g_2, g_4).(f \times h)^k(V)) \neq Ø$. So we have $Ø \neq ((g_1, g_3).(f \times h)^k(U_1 \times Y)) \cap ((g_2, g_4).(f \times h)^k(V_1 \times Y)) = (g_1 . f^k(U_1) \times g_3 . h^k(Y)) \cap (g_2 . f^k(V_1) \times g_4 . hk(Y)) = (g_1 . fk(U_1) \cap g_2 . f^k(V_1)) \times (g_3 . h^k(Y) \cap g_4 . h^k(Y))$, thus, $g_1 . f^k(U_1) \cap g_2 . f^k(V_1) \neq Ø$, then f is G_1-weakly blending. By the same way, we can prove h is G_2-weakly blending.

3. 6. Theorem

Let X be a G_1-space and Y be a G_2-space. $f:X\to X$ and $h:Y\to Y$ are maps. Then f is G_1-strongly blending and h is G_2-strongly blending if and only if $f\times h:X\times Y\to X\times Y$ is $G_1\times G_2$-strongly blending.

Proof:

It is similar to the proof of (3.5).

3. 7. Theorem

Let $f:X\to X$ be a continuous action on X. If f is G-strongly blending and has G-periodic points, then f satisfy G-Touhey property.

Proof:

Let U and V be nonempty open subsets of X. So there exist $k > 0$ and $g_1, g_2 \in G$ such that $g_1. f^k(U) \cap g_2. f^k(V)$ contains open set, therefore, there exists an open set $W \subset X$ such that $W \subset g_1. f^k(U) \cap g_2. f^k(V)$. Let $V_1 = g. f^{-k}(W) \cap V$, we note $g. f^{-k}(W)$ is open since W is open set and f continuous map, thus V_1 is open. Since the set of G-periodic points is dense, then there exists a G-periodic point p of period m such that $g. f^k(p) \in V$, so there is $y \in U$ such that $g. f^k(y) = g. f^k(p)$. Therefore, $g. f^m(y) = g. f^{m-k}(f^k(y)) = g. f^{m-k}(f^k(p)) = g. f^m(p) = p$, so $p \in f^m(U) \cap V \neq \emptyset$, then $p \in g. f^m(V)$, and consequently f satisfies G-Touhey property.

3. 8. Lemma

Let X and Y be G_1 and G_2 spaces respectively. $f:X\to X$ and $h:Y\to Y$ are maps. The set of $G_1\times G_2$-periodic points are dense of $f\times h$ if and only if for f the sets of G_1-periodic points in X are dense and for h the sets of G_2-periodic points in Y are dense.

Proof:

We will prove the set of $G_1 \times G_2$-periodic points of $f \times h$ is dense in $X \times Y$. Let W be a nonempty open set of $X \times Y$, then there are a two nonempty open sets U of X and V of Y such that $U \times V \subseteq W$. By density of G_1-periodic points there exist: $x \in U$ and $n \in N$ with $g_1. f^n(x) = (x)$ for some $g_1 \in G_1$, also by density of G_2-periodic points there exist $y \in V$ and $m \in N$ with $g_2. f^m(y) = y$ for some $g_2 \in G_2$. Now, for $(x, y) \in$ W, $(g_1, g_2) \in G_1 \times G_2$ and $k = nm$, we have $(g_1, g_2). (f\times h)(x, y) = (g_1, g_2). (f^k(x), h^k(y)) = (g_1. f^k(x), g_2. h^k(y)) = (x, y)$. So that (x, y) is $G_1 \times G_2$-periodic point of $f\times h$.

Thus, we get that the set of $G_1 \times G_2$-periodic points of $f \times h$ is dense in $X \times Y$. Conversely, suppose that U, V are two nonempty open sub sets of X and Y respectively, then $U \times V$ is nonempty open subset of $X \times Y$. Since the set of $G_1 \times G_2$-periodic points of $f \times h$ is dense in $X \times Y$ then there exist $(x, y) \in U \times V$ and $n \in N$ such that $(g_1, g_2). (f \times h)\, n\, (x, y) = (g_1. f^n(x), g_2. h^n(y)) = (x, y)$ for some $(g_1, g_2) \in G_1 \times G_2$. So that from this equality, we get that $g_1. f^n(x) = x$ for $x \in U$ and $g_1 \in G_1$, also $g_2. h^n(y) = y$ for $y \in V$ and $g_2 \in G_2$. Thus, the set of G_1-periodic points of f is dense in X and the set of G_2-periodic points of h is dense in Y.

3. 9. Proposition

A map $f: X\to X$ satisfy G-Touhey property on X if and only if f is G-transitive and the G-periodic points of f are dense in X.

Proof:

Let f be G-chaotic on X then every pair of nonempty open sets shares a G-periodic orbit. In particular, every nonempty open set must contains a G-periodic point so the G-periodic points of f are dense in X. By Definition of G-Touhey, for each pair of subsets there are $n > 0$ and $g \in G$ such that $g.f_n(U) \cap V$ shares a G-periodic orbit so the intersection is not empty, then f is G-transitive.

Conversely, we assume that f is G-transitive and has a dense set of G-periodic points. Let U and V be any pair of nonempty open subsets of X. Since f is G-transitive then there exist $u \in U$, $g \in G$ and there is $k \in N$ such that $g. f^k(u) \in V$. We can define $W = g. f^{-k}(V) \cap U$. since the intersection of two open subsets of X and u is an element of both of them then W is also open and nonempty. W has the property that $g. f^k(W) \subset V$; but the G-periodic points of f are assumed to be dense in X, so the nonempty open set W must contain a G-periodic point $p \in W \subset U$ with the property that $g. f^k(p) \in g. f^k(W) \subset V$. So, we get that f is G-chaotic.

3. 10. Theorem

Let X be a G_1-space and Y be a G_2-space, $f:X \rightarrow X$ and $h:Y \rightarrow Y$ be continuous maps, if f is G_1-strongly blending and has dense G_1-periodic points, h is G_2-strongly blending and has dense $G2$- periodic points then $f \times h$ is $G_1 \times G_2$-chaotic.

Proof:

Since f is G_1-strongly blending and h is G_2-strongly blending, then by Theorem (3.6) $f \times h$ is $G_1 \times G_2$-strongly blending. Also since f has dense G_1-periodic point and h has dense $G2$-periodic point, then by Lemma (3.8) $f \times h$ has dense $G_1 \times G_2$-periodic point. Then by Theorem (3.7) $f \times h$ satisfy G-Touhey property. Therefore, by Proposition (3.9) $f \times g$ is $G_1 \times G_2$-chaotic.

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